A POSSIBLE ANOMALOUS WAVE TRANSFORMATION IN PLASMA

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It is a familiar fact that in the approximation of geometrical optics the normal oscillations in a weakly inhomogeneous medium are independent. The approximation of geometrical optics is, however, violated close to those points where either the wave vector k(x) becomes zero (reversal point), or where the wave vectors corresponding to the different types of oscillation coincide (points of intersection of the solutions). In the immediate neighborhood of these points separation into normal oscillations is no longer possible, which in the case of "points of intersection of the solutions" leads to the possible appearance, in addition to the wave incident from infinity, of another new wave with different dispersive properties.

However, cases so far considered of the "birth" of a new wave due to the passage of an incident wave with such properties have led to an exponentially small transformation coefficient, and also to the absence of reflected waves (see, for example, [1]; ibid. bibliography).

In [2] the rules for going around points of "intersection" of the solutions were obtained for a system of two coupled oscillators, which also led to the appearance of the reflected solutions

$$\exp\left(\int \omega(t)\,dt\right) \to \exp\left(-\int \omega(t)\,dt\right),$$

where $\omega(t)$ is the normal frequency of the oscillator and varies slowly with time. The transformation and reflection coefficients here turned out to be exponentially small.

The problem of wave propagation in a medium is the exact mathematical equivalent of the problem of the oscillations of two coupled oscillators (only instead of considering $\omega(t)$ as a slowly varying function of time we must consider k(x) as a slowly varying function of the coordinate). Thus the results [2] may be transferred to the case of wave transformation.

In what follows we consider the appearance of reflected waves when the solutions "intersect," and also the appearance of transmitted waves with a large transformation coefficient.

1. We shall represent k_1 in the form

$$k_1 = {}^1/_2\,(k_1 + k_2) + {}^1/_2\,(k_1 - k_2). \qquad (1.1)$$

Here $k_1(x)$, $k_2(x)$ are the wave vectors which coincide at some point in the complex x-plane.

It is clear that if the expression $k_1 - k_2$ is many-valued in the neighborhood of points where the solutions "intersect," then it is possible for a wave to appear, only with different dispersion properties $k_1 \to \to k_2$ (since only the sign of the difference $k_1 - k_2$ can change).

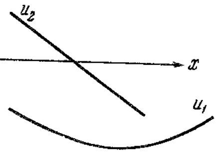

Fig. 1

However, if $k_1 + k_2$ is also a multi-valued function, then the transition $k_1 \to -k_1$, for example, is possible; this corresponds to the appearance of reflected waves. It should be stressed at this point that for the appearance of reflected waves it is a necessary, but not sufficient condition that $k_1 - k_2$, $k_1 + k_2$ should be multi-valued, since the pattern according to which the level lines for

$$\int (k_1 - k_2)\,dx, \qquad \int (k_1 + k_2)\,dx$$

are disposed relative to the real axis is also of importance.

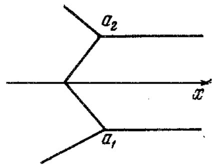

Fig. 2

At least there appear specifically new conditions for going around singular points compared with a second-order differential equation, specifically in the case where $k_1 - k_2$ and $k_1 + k_2$ become multi-valued simultaneously (as is clear from (1.1), the different components from which k_1, for example, is composed, have their own system of level lines).

We shall here try to determine the possibility of the "birth," in the region where the solutions "intersect," of new waves ($k_1 \to k_2$) with a coefficient of the order of unity.

If a large transformation is to be achieved, then conditions must be created which do not allow the incident wave to be transmitted to some region where, however, the second of the mutually coupled waves may propagate freely. We note that in [2], as a result of the way in which the problem was posed, the coefficients of the equations vanished nowhere, and the coupled oscillations existed over the whole real axis. This, in its turn, led to a pattern of level lines for

$$\int (k_1 + k_2)\,dx, \qquad \int (k_1 - k_2)\,dx$$

exactly similar to the pattern of level lines for $\int p\,dx$ in quantum mechanics in the case of reflection where the barrier energy is exceeded (see [3], here p is the impulse of the particle). We recall that in this latter case an exponentially small reflection coefficient is also obtained.

In what follows, in order to be specific, we shall consider the differential equation

$$\alpha\beta^2\varphi^{IV} - u_2(x)\,\beta\varphi'' + u_1(x)\,\varphi = 0 \quad (\beta = \lambda/R) \qquad (1.2)$$

with two small parameters α and β. Here λ is the wavelength of the oscillation, and R is the characteristic dimension of the irregularity; the second small parameter α is connected with the concrete physical situation; the discussion given here applies, in particular, to the case where $\alpha = 1$, $u_1 \sim 1$, $u_2 \sim \sim 1$, with the exception of the points where they may become zero.

We shall avail ourselves of the following representation for (1.2):

$$k_{2,1} = \frac{1}{\sqrt{2\beta}}\left[\left(\frac{u_2}{2\alpha} + \left(\frac{u_1}{\alpha}\right)^{1/2}\right)^{1/2} \pm \left(\frac{u_2}{2\alpha} - \left(\frac{u_1}{\alpha}\right)\right)^{1/2}\right]. \quad (1.3)$$

We shall investigate the behavior of $k_{2,1}$ close to the point where $u_2 = 0$ (Fig. 1). It follows from (1.3) that in the region where $x < 0$ oscillating solutions exist with wave vector k_1; correspondingly, in the region where $x > 0$ there are solutions with wave vector k_2 (we take the origin at the point $u_2 = 0$). Further, writing $u_2 = ux$, $u_1 = u_{10}$ (in the neighborhood of the point $u_2 = 0$ the specific behavior of u_1 is immaterial), we see from (1.3) that not only is $k_1 - k_2$ a multi-valued function, but also $k_1 + k_2$.

The pattern according to which the level lines

$$\int (k_1 - k_2)\, dx, \qquad \int (k_1 + k_2)\, dx$$

are disposed relative to the real axis is now quite different from the pattern of level lines for these expressions in the case similar to reflection in quantum mechanics where the barrier energy is exceeded (see Fig. 2; the branch points for $k_1 - k_2$ and $k_1 + k_2$ are indicated by a_1 and a_2, respectively).

We also draw attention to the fact that for concrete applications of Eq. (1.2) to the question of transformation of oscillations $u_1 < 0$ at the point where $u_2 = 0$, as depicted in Fig. 1. To convince ourselves of this we consider the system of two coupled oscillators

$$\begin{aligned} \alpha x'' + \omega_1^2(t)\, x &= v(t)\, y, \\ y'' + \omega_2^2(t)\, y &= v(t)\, x \end{aligned} \quad (1.4)$$

to which the following fourth-order differential equation corresponds

$$\alpha x^{IV} + (\omega_1^2 + \alpha\omega_2^2)\, x'' - (v^2 - \omega_1^2\omega_2^2)\, x = 0 \quad (1.5)$$

when the coefficients vary slowly.

It is clear that

$$u_2 = -(\omega_1^2 + \alpha\omega_2^2), \quad u_1 = -(v^2 - \omega_1^2\omega_2^2);$$

$$u_1 < 0 \quad \text{for} \quad u_2 = 0.$$

It has been pointed out above that oscillating solutions with different dispersion properties exist on different sides of the point $u_2 = 0$, and so a large mutual wave transformation is to be expected here. To be finally convinced of this we consider the coupling between the solution of Eq. (1.2) on either side of the point $u_2 = 0$. Making the substitution $x = \beta y$, we reduce (1.2) in the neighborhood of the point $u_2 = 0$ to the form

$$\varphi^{IV} - \lambda_+^2 (uy\varphi'' + u_{10}\varphi) = 0 \qquad (\lambda_+^2 = \beta^2/\alpha). \quad (1.6)$$

The properties of the solutions of (1.6) for $\lambda_+ \gg 1$ were investigated by Laplace's method in [4], and for $\lambda_+ \ll 1$ by the phase integral method in [5, 6] (in this case the distance between the points a_1 and a_2 is large compared with the wavelength of the intersecting solutions, and we may go around each singular point separately).

We note that it follows from the analysis of solutions obtained in [4, 5] that the asymptotic solutions are similar for $\lambda_+ \gg 1$ and $\lambda_+ \ll 1$. In what follows we shall need only the asymptotic form of one of the solutions obtained in [4] for large y,

$$\varphi = \pi i y^{1/2} u_{10}^{-1/2} H_1^{(1)} (2u_{10}^{1/2} y^{1/2}) \quad (y < 0)$$

$$\varphi = \sqrt{\pi}\lambda_+^{-2/3} (3/_2\xi)^{-5/6} e^{-\xi} +$$

$$+ \pi i y^{1/2} u_{10}^{-1/2} H_1^{(1)} (2u_{10}^{1/2} y^{1/2}) \quad (y > 0)$$

$$\xi = i\lambda_+^{2/3} y^{3/2}. \quad (1.7)$$

We see from (1.7) that in the case when the form of u_2 and u_1 is similar to that depicted in Fig. 1, the solution which has wave vector k_1 for $y < 0$ passes, for $y > 0$, into the solution with wave vector k_2 (it is not difficult to see that $H_1^{(1)}(2u_{10}^{1/2} y^{1/2})$ is exponentially damped for large $y > 0$).

2. We will now consider some concrete applications.

In [7] consideration was given to the question of the mutual transformation of fast and slow magnetosonic waves in the case when only the expression $(k_1 - k_2)$ is multi-valued. The equation for these waves obtained in [7] has the form

$$\varphi^{IV} + \left(\frac{\omega^2}{v_A^2} + \frac{\omega^2}{S^2} - k_y^2\right)\varphi'' +$$

$$+ \left[\frac{\omega^2}{v_A^2}\left(\frac{\omega^2}{S^2} - k_y^2\right) - \frac{\omega^2 k_y^2}{S^2}\right]\varphi = 0$$

$$v_A = \frac{H_0}{\sqrt{4\pi\varphi_0(z)}}. \quad (2.1)$$

Here S is the velocity of sound, v_A is the Alfvén velocity, $\varphi_0(z)$ is the density of the medium which is nonuniform along the magnetic field $\mathbf{H_0}$ ($\mathbf{H_0}$ is directed along the z axis).

It is not difficult to see that if there exist points z_0 in the medium for which the condition

$$k_y^2 = \frac{\omega^2}{v_A^2(z_0)} + \frac{\omega^2}{S^2} \quad (2.2)$$

is fulfilled, then all the considerations set forth here, and, in particular, formula (1.7), are applicable to (2.1). We note that the question of the anomalous mutual transformation of magnetohydrodynamic waves may be of particular significance in constructing a theory of heating of the chromosphere [8]. We shall consider the question of the conversion of a plasma wave to an electromagnetic wave, which was treated in [9]. The system of equations for this case has the following form (see [1]):

$$\frac{d^2w}{dz^2} - \frac{d\varepsilon}{dz}\frac{1}{\varepsilon - \alpha_1^2\beta_T^2}\left(1 + \frac{\alpha_1^2\beta_T^2}{1-\varepsilon}\right)\frac{dw}{dz} +$$

$$+ k_0^2(\varepsilon - \alpha_1^2)w = \frac{\alpha_1\beta_T^2}{(1-\varepsilon)(\varepsilon - \alpha_1^2\beta_T^2)}\frac{d\varepsilon}{dz}\frac{du}{dz}, \qquad (2.3)$$

$$\beta_T^2\frac{d^2u_E}{dz^2} + \frac{\varepsilon\beta_T^2}{(1-\varepsilon)(\varepsilon - \alpha_1^2\beta_T^2)}\frac{d\varepsilon}{dz}\frac{du_E}{dz} +$$

$$+ k_0^2(\varepsilon - \alpha_1^2\beta_T^2)u_E = -\frac{\alpha_1\beta_T^2}{\varepsilon - \alpha_1^2\beta_T^2}\frac{d\varepsilon/dz}{1-\varepsilon}\frac{dw}{dz} - k_0^2\alpha_1 w$$

$$k_0 = \frac{\omega}{c}, \quad \beta_T = \frac{v_T}{c}, \quad \varepsilon = 1 - \frac{4\pi e^2 N}{m\omega^2} \quad (v_T \ll c). \qquad (2.4)$$

Here w is the amplitude of the component on the x axis of the magnetic field of normal oscillations, u_E is the amplitude of the component on the z axis of the electric field of normal oscillations, α is the sine of the angle of incidence of the plasma wave on a plasma layer with density N varying in the z direction, k_0 is the wave vector of the electromagnetic wave, β_T is a small parameter, v_T is the thermal velocity of the electrons, e is the electronic charge, m is the electronic mass. For simplicity, we neglect wave absorption. As in [9], we approximate $\varepsilon(z)$ by the linear function

$$\varepsilon(z) = -z \operatorname{grad} \varepsilon \quad (\operatorname{grad} \varepsilon \sim R^{-1}). \qquad (2.5)$$

If R is large enough, then the zeros of u_2, u_1 in the fourth-order differential equation equivalent to (2.3) and (2.4) are situated at the points

$$z_0 = -2\alpha_1^2\beta_T^2 R, \qquad z_1 = -R\alpha_1^2, \qquad z_2 = -R\alpha_1^2\beta_T^2,$$

respectively, and the graph of u_2, u_1 has the form depicted in Fig. 3.

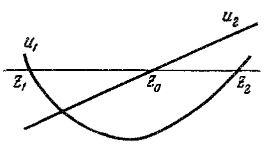

Fig. 3

Moreover, Eqs. (2.3) and (2.4) have a singularity at the point $z_2 = -R\alpha_1^2\beta_T^2$. If

$$\omega R\alpha_1^3\beta_T^2 c^{-1} \gg 1, \qquad (2.6)$$

then, in passing around the point $u_2 = 0$, we may omit taking the above-mentioned singularity into account.

We note that condition (2.6) is also equivalent to the fact that there are many wavelengths of "mode" k_1 (the wavelength of which $\lambda_1 \sim c/\omega\alpha_1$) packed into the region z_0z_2. It is then easy to see that in the neighborhood of the point $u_2 = 0$ the system (2.3), (2.4) is similar to Eq. (1.2) [the presence of the first derivative in the equation introduces no significant variations in the rules for going around $u_2 = 0$ (see [10])].

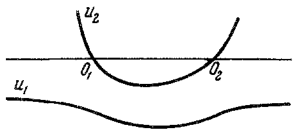

Fig. 4

Oscillating solutions of the plasma "mode" (with wavelength $\lambda_2 \sim r_d$, where r_d is the Debye radius of the plasma) exist in the region $z < z_0$. If this normal wave is incident from the left on the neighborhood of the point z_0, then, according to (1.7), for $z > z_0$ it passes completely into another normal wave with wave vector k_2, which could be recorded by a receiver situated in the region z_0z_2. In the case investigated in [9], however, the escape of electromagnetic radiation to the exterior of the sun was considered, i.e., in the direction $z \ll z_0$ (the density of the solar plasma increases in the direction $z \to \infty$). In this case electromagnetic radiation reflected from the point z_2 is damped in the region z_1z_0, and oscillates further on. The presence of a barrier for electromagnetic radiation results in only a small amount of it emerging from the solar plasma.

We note, however, that a large transformation is possible between the ordinary and extraordinary waves. Thus it may be shown that anomalous transformation is possible in a rarefied plasma if the Larmor frequency corresponding to the magnetic field component in the direction of wave propagation is of the order of the frequency of these waves.

Besides being of intrinsic interest, the question of wave transformation may be of importance in the problem of plasma stability. To demonstrate this we consider the following example. Let the behavior of the coefficients u_1 and u_2 be similar to that depicted in Fig. 4. In this case one of the "coupled" waves (with wave vector k_1) oscillates at infinity, and the other is damped. Let a localized disturbance ("packet"), which increases with time and is formed by waves with wave vector k_2, arise in the central position of the region O_1O_2. In this case if the rate at which energy escapes to infinity on account of the transformation into another type of wave exceeds the rate at which energy passes into the disturbance from the instability sources, then the instability does not develop and the plasma itself may now serve as a generator of oscillations which go off to infinity.

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